$$\lambda \leq 1 - \left\{ \frac{1-(A_1+\sqrt{2}A_2)}{0.2(A_1+\sqrt{2}A_2)} \right\} \tag{22}$$

$$\lambda \leq 1 - \left\{ \frac{4A_2 - 3(2A_1A_2+\sqrt{2}A_1^2)}{0.6(2A_1A_2+\sqrt{2}A_1^2)} \right\} \tag{23}$$

$$\lambda \leq 1 - \left\{ \frac{2-(A_1+\sqrt{2}A_2)}{0.2(A_1+\sqrt{2}A_2)} \right\} \tag{24}$$

$$\lambda \leq 1 - \left\{ \frac{0.10 - A_i}{0.02} \right\}; \quad i=1,2 \tag{25}$$

The solution of this problem is given by $\lambda^* = 0.5226$, $A_1^* = 0.60856$, $A_2^* = 0.86065$ with $f^* = 2.58192$. The constraints of Eqs. (20), (21), and (24) are active at the optimum solution. Notice that the fuzzy formulation has increased the number of design variables by one.

The fuzzy optimum solution has the following interpretation. When the range of the permissible stress in member 1 (σ_1) is stated as 20–24, it implies that a values of $\sigma_1 = 20$ has the maximum degree of satisfaction ($\mu = 1$) and a value of $\sigma_1 = 24$ has a minimum degree of satisfaction ($\mu = 0$). Since the membership function of σ_1 is assumed to have inclined line boundaries, the degree of satisfaction increases linearly from 0 to 1 as σ_1 decreases from 24 to 20. The permissible ranges on all other response parameters, including the objective function, have similar meanings. The fuzzy optimum solution indicates that the maximum level of satisfaction (degree of membership) that can be achieved in the presence of the stated fuzziness in the objective function and constraints is 0.5226.

Conclusion

The description and the optimization of structures containing fuzzy information has been presented. A method of solving fuzzy optimization problems using ordinary nonlinear programming techniques was presented along with a numerical example. The procedures outlined are expected to be useful in situations where doubt arises concerning exactness of concepts, correctness of statements and judgments, degrees of credibility, etc., which are essential for the definition of a crisp design optimization problem. The possible application of fuzzy set theory in the analysis of structural failures and the development of expert systems for structural design are also indicated.

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Iterative Study for Three-Dimensional Finite-Element Stress Analysis

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Introduction

THE use of three-dimensional finite-element models based on three-dimensional elasticity theory for the analysis of layered laminates is increasing,[1] but this often results in a large numerical model. The aim of this work is to reduce the total number of degrees of freedom by selecting three-dimensional elements in a coarse mesh.

The Loubignac method[2,3] improves the accuracy of both stresses and displacements in finite-element stress analysis. It produces a stress field that is continuous across interelement boundaries by using average nodal stresses. However, this method cannot be used for problems such as multilayered plates[2] which consist of bonded dissimilar layers because the in-plane stresses are not continuous across the interface of dissimilar materials. A modified algorithm is presented and applied in layered nonhomogeneous structures.

Iterative Formulation

The standard finite-element method in structural mechanics gives a set of equations

$$[K]\{\delta\}_1 = \{R\} \tag{1}$$

where $[K]$ is the structural stiffness matrix, $\{R\}$ is the vector of nodal forces, and $\{\delta\}_1$ is the nodal displacement vector. By using Eq. (1), we can determine the stresses $\{\sigma\}$ in each element by means of the relations

$$\{\sigma\} = [D][B]\{u\}_1 \tag{2}$$

where $[B]$ is the strain displacement matrix, $\{u\}_1$ is a vector of the element nodal displacements which is a component of $\{\delta\}_1$, and $[D]$ is the elastic constitutive matrix. Because of the displacement-based finite-element formulation, the stresses obtained are discontinuous across interelement boundaries: this discontinuity will decrease with mesh refinement. Loubignac et al.[2,3] computed the average stresses $\{\bar{\sigma}_N\}$ at a common node across interelement boundaries. Within an element, let the stresses $\{\bar{\sigma}\}$ be interpolated from the average nodal stresses $\{\bar{\sigma}_N\}$ in the same way that displacements are interpolated from nodal degrees of freedom (DOF) by the shape function $[N]$. Thus,

$$\{\bar{\sigma}\} = [N](\bar{\sigma}_N) \tag{3}$$

When applying the Loubignac iteration in a structure, the determination of the nodal stresses across dissimilar materials by averaging must be avoided. In order to solve more general structural problems, the Loubignac method should be modified. If the orthogonal curvilinear coordinates are r, s, and t, then the stresses and strains in a solid are

$$\{\sigma_r, \sigma_s, \sigma_t, \tau_{st}, \tau_{tr}, \tau_{rs}\}$$

and

$$\{\epsilon_r, \epsilon_s, \epsilon_t, \gamma_{st}, \gamma_{tr}, \gamma_{rs}\}$$

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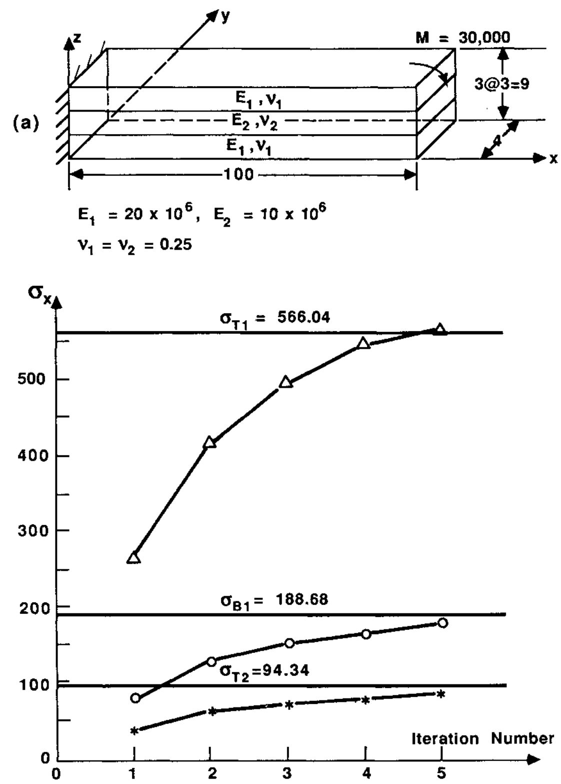

Fig. 1 Three-layered cantilever solid beam: a) geometry and material properties b) results of σ_x at $x=66$, $y=2$.

respectively. Assuming that $\{e_t\}$ is a unit vector that is normal to an interface, then the stresses σ_t, τ_{st}, and τ_{tr} are continuous across that interface. Furthermore, the strain components ϵ_r, ϵ_s, and τ_{rs} are also continuous, while the stress components σ_r, σ_s, and τ_{rs} may not be continuous. In view of this observation, a new continuous nodal mixed field, represented by

$$\{\bar{\sigma}_N^*\} = \{\epsilon_r, \epsilon_s, \sigma_t, \tau_{st}, \tau_{tr}, \epsilon_{rs}\} \tag{4}$$

is determined by averaging after the corresponding terms are obtained from Eqs. (1) and (2) and the related strain-displacement relation within an element. In most problems, the reference axes x, y, and z are equivalent to the orthogonal coordinates γ, s, and t. If they are not coincident to each other, a coordinate transformation is applied.[4]

The average (mean or weighted average) nodal mixed field $\{\bar{\sigma}^*\}$ is interpolated from the $\{\bar{\sigma}_N^*\}$ with a modified shape function $[\bar{N}]$ which may or may not be the same as the standard shape function $[N]$. Thus,

$$\{\bar{\sigma}^*\} = [\bar{N}]\{\bar{\sigma}_N^*\} \tag{5}$$

From the element stress-strain relations, we can compute the stress $\{\bar{\sigma}\}$ or strain $\{\bar{\epsilon}\}$ fields from $\{\bar{\sigma}^*\}$ and enforce the stress or strain boundary conditions as necessary.

Next we consider $\{\bar{\sigma}\}$ as an initial stress and impose the condition that the sum over all elements should balance the applied load vector. The first iterative nodal vector $\{Q\}_1$ that corresponds to the stress $\{\bar{\sigma}\}$ is

$$\{Q\}_1 = \sum_{\text{elements}} \int [B]_V^T(\bar{\sigma})\mathrm{d}v \tag{6}$$

If Eq. (1) yields the exact solution, where $\{\bar{\sigma}\}$ is exact, then $\{R\} = \{Q\}_1$; otherwise $\{R\} - \{Q\}_1$ is a load imbalance indicator which can be used to improve the solution. The various steps in this procedure are

$$\{\delta\}_1 = [K]^{-1}\{R\} \tag{7}$$

$$\{\Delta\delta\}_1 = [K]^{-1}(\{R\} - \{Q\}_1) \tag{8}$$

Then, the new displacement vector is

$$\{\delta\}_2 = \{\delta\}_1 + \{\Delta\delta\}_1$$

$$= [K]^{-1}(2\{R\} - \{Q\}_1) \tag{9}$$

Hence, a general expression can be developed in the form

$$\{\delta\}_i = [K]^{-1}(i\{R\} - \{Q\}_1 - \{Q\}_2 - \ldots\ldots - \{Q\}_{i-1})$$

$$= [K]^{-1}\{R\}_i \tag{10}$$

where i is the iteration number and $\{R_i\}$ is a perturbed load vector. Using Eqs. (2), (4), and (6) yields the new $\{Q\}_i$. In Eq. (10), $[K]$ is formed and constructed only once. Thus, the need for the reconstruction and triangularization of the stiffness matrix after each iteration is eliminated. Normally, the scheme converges in a few iterations. In the next section, our iterative scheme and the classical approach are compared in the analysis of two solid cantilever beams.

Numerical Tests

In the example that follows, the performance of the iterative formulation is demonstrated by comparing it to analytical results. The integration formulas used in the stiffness calculation for the three-dimensional elements are: a $2\times2\times2$ complete rule for eight-node trillinear brick elements, special 13-or-14-point rules for 20- and 21-node solid elements[5], and a $3\times3\times3$ complete rule for the 27-node Lagrange element are used.

The test end load is applied according to the consistent force formulation. For the undistorted case, eight-node and 20-node solid isoparametric elements are used. We omit the distorted and skew-distorted cases in this Note. Their development is, however, documented in Ref. 6.

The example problem which appears in Fig. 1 is that of a symmetric three-layered nonhomogeneous cantilever solid beam loaded with a bending moment $M=30{,}000$ at the free end. In this problem, the free-edge effect is not considered. The horizontal stresses σ_x are discontinuous across the dissimilar material interface. The analytical results for σ_x, based on the classical lamination theory, are $\sigma_{T1}=566.04$, $\sigma_{B1}=188.68$, and $\sigma_{T2}=94.34$, respectively, where σ_{T1} and σ_{B1} are the stresses at the top and bottom fiber of the top layer and σ_{T2} is the stress at the top fiber of the middle layer. The finite-element model incorporates $6\times1\times1$ eight-node brick elements in each layer. Numerical results for σ_x at $x=66$, $y=2$ are shown in Fig. 1 after five iterations.

Conclusions

Three-dimensional finite models using a coarse mesh system have been demonstrated for the problems involving dissimilar materials. In averaging across an existing node, it was found that the mathematical stress singularity[7] needed further study because the stresses were not defined at those nodes. Since there are no changes to the primary finite-element solutions, the iterative part can be implemented as a postprocessing subroutine in any existing program which, based on the assumed-displacement model, can handle the anisotropic materials without altering the main body of the code.

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Error Bounds for Eigenvalues of Unconstrained Structures

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Introduction

THE methods of Krylov-Bogoliubov and Kato-Temple (to be referred to as the Krylov and Kato methods, respectively) for determination of the bounds on natural frequencies are generalized in this Note to the case of an unconstrained mechanical system. The eigenfrequencies ω and eigenvectors x of free vibrations of a discretized elastic system are defined as the solution of the equation

$$Kx = \omega^2 Mx \tag{1}$$

where the stiffness matrix K and the mass matrix M are symmetric and positive definite or semidefinite. Given a certain approximation x_0 of an eigenvector x, the Krylov[1] method supplies the following estimate of the eigenvalue:

$$\frac{1}{\rho+\sqrt{\sigma^2-\rho^2}} \le \omega^2 \le \frac{1}{\rho-\sqrt{\sigma^2-\rho^2}} \tag{2}$$

where

$$\rho = x_0^T M x_0 / x_0^T K x_0 \tag{3}$$

$$\sigma^2 = x_0^T M K^{-1} M x_0 / x_0^T K x_0 \tag{4}$$

Assume that the eigenvalues are numbered in ascending order, as

$$\omega_1^2 \le \omega_2^2 \le \ldots \le \omega_n^2$$

and values μ and ν are known such that $1/\mu$ is an upper bound to ω_{k-1}^2 and $1/\nu$ is a lower bound to ω_{k+1}^2

$$1/\omega_{k+1}^2 \le \nu < \rho < \mu \le 1/\omega_{k-1}^2 \tag{5}$$

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Then, the Kato[2] formula can be applied for determination of bounds on eigenvalue ω_k^2 as

$$\frac{1}{\rho+(\sigma^2-\rho^2)/(\rho-\nu)} \le \omega_k^2 \le \frac{1}{\rho+(\sigma^2-\rho^2)/(\rho-\mu)} \tag{6}$$

The Krylov method is usually applied for computation of values μ and ν, which are then used in the Kato formulas, often supplying more restrictive bounds than those of the Krylov method alone. For example, these methods were applied in Refs. 3 and 4 to error estimates of the eigenvalue approximations obtained through the elimination of variables. They may serve as well as reliable termination criteria in iteration methods determining eigenvalues (especially the subspace iteration method).[5]

Since Eq. (4) contains the inverse of the stiffness matrix K, the above formulas can be used only for constrained systems for which K is nonsingular. When a structure and/or its part can undergo displacements as a rigid body (without storing potential deformation energy), its stiffness matrix is singular. In the case of such an unconstrained structure, there are two possibilities of determining error bounds for eigenfrequencies.

The first one can be applied only if the mass matrix M is nonsingular. As the eigenproblem,

$$Mx = (1/\omega^2)Kx \tag{7}$$

is equivalent to Eq. (1) for $\omega \neq 0$, each nonzero eigenfrequency can be estimated by the use of the Krylov and Kato formulas, in which ω is replaced by $1/\omega$ and the roles of K and M are interchanged.

The second method of removing the singularity of K consists in eigenvalue shifting, that is, the addition of the vector $\alpha Mx (\alpha > 0)$ to both sides of Eq. (1) leads to the eigenproblem

$$(K+\alpha M)x = (\omega^2+\alpha)Mx \tag{8}$$

In order to apply the methods of Krylov and Kato, the nonsingular matrix $K+\alpha M$ must be inverted.

The aim of this Note is to present an alternative method that uses the flexibility matrix of the system under consideration, one in which the statically determinable constraints (removing all the rigid-body modes) are imposed.

Eigenproblem Formulation for Unconstrained Structure

Suppose that the system has n degrees of freedom, including r rigid-body degrees. Then there exists a matrix R of rigid-body modes, containing r linearly independent columns and satisfying the equation

$$KR = 0 \tag{9}$$

It is easy to notice that the rigid-body modes are the solutions of the eigenproblem represented by Eq. (1) for $\omega = 0$. Besides, there exist $m = n - r$ linearly independent eigenvectors $x_1, x_2, \ldots, x_m$ associated with the nonzero eigenfrequencies and called deformation modes

$$Kx_i = \omega_i^2 Mx_i \quad (i = 1,2,\ldots,m) \tag{10}$$

They are known to have the orthogonality properties, which can be written after suitable normalization as

$$x_i^T K x_j = \delta_{ij} \text{ and } x_i^T M x_j = \lambda_i \delta_{ij} \quad (i,j = 1,\ldots,m) \tag{11}$$

where δ_{ij} is the Kronecker's delta and

$$\lambda_i = 1/\omega_i^2 \quad (\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m) \tag{12}$$